Mitigation and Solar Radiation Management in Climate Change Policies

1 Introduction

The theory of global warming has received much attention in the last decades. Mathematical modelling of climate is of interest because of the concern about the planet's climatic response to man's activities. Technological evolution and the development of numerical weather prediction models have made the simulation of climate feasible. Global scale collection and analysis of observations have provided a base for developing and verifying models. Studies of the earth's climate suggest that the present state of global warming is due to adverse changes through man's activities. As humans demand more and more from the earth's resources the growing human population becomes less able to cope with climatic change; the result of this fact is that the forecasting of future climate becomes increasingly important. The science of global warming is of great importance due to the predictions of climatologists about the increase of temperature in the coming years. There are many aspects to the way that man can interfere with the climate and reverse the damage to the earth's climate that has been brought about in the past years but the basic options are mitigation and geoengineering.

The theory of energy-balance is based on the idea of global radiative heat balance. In radiative equilibrium the rate at which solar radiation is absorbed matches the rate at which infrared radiation is emitted. The main aim is to reduce global average temperature by controlling the incoming solar radiation. Scientists have proposed models for the control of incoming solar radiation which use an ice feedback mechanism in order to increase earth's albedo. In these models they study ice sheet latitude as a function of solar constant and they have proved that the present climate as well as an ice-covered earth are stable while an intermediate solution is unstable for small perturbations away from equilibrium $[1-3]$.

Climatologists and other scientists have warned for more than half a century that the accumulation of CO2 and other GHGs in the atmosphere is leading to global warming and other significant climatic, ecological, and societal changes. The difficulty in reducing emissions is that it is a public good, meaning that no single nation can capture for itself a substantial part of the benefits from its own emission reductions. In the attempt to trace the impact of policies to limit emissions on global temperature Nordhaus [4], C. Criado, S. Valente, T. Stengos [22], A. Xepapadeas [23] have developed models which study the relative advantages and disadvantages of policies seeking to limit atmospheric concentrations of carbon dioxide (CO_2) and other greenhouse gases ($GHGs$) in order to achieve a reduction of global temperature.

In the last few years it appears to be technically feasible to engineer an increase in the planet's albedo, as a means to offset the warming caused by carbon dioxide (CO_2) and other greenhouse gases emissions, through solar ra-

diation management [6], [25]. Solar radiation management is also referred as geoengineering. In this paper we examine the specific geoengineering proposal that implies spreading very small reflective particles to shade the earth from the sun, pumping sulphur dioxide into the stratosphere and loft sea salt into the atmosphere, or building a sunscreen in space. The main advantages of geoengineering that makes it attractive for managing climate risk are that it is quick and cheap. The option of limiting global warming through mitigation, that is a reduction of emissions, needs time until we have significant results in the confrontation with climate changes. On the other hand, geoengineering proposes ways which have instant impacts on the climate. According to Robock [5] the techniques of geoengineering are much less expensive than the time-consuming techniques of mitigation and reduction of emissions.

However, there are many disadvantages in the techniques that geoengineering suggests : solar radiation management cannot exactly compensate for the emission of CO_2 which causes climate change, there will be less sun radiation available for solar power systems, there will be negative effects on plants and these techniques cannot adjust temperature at the desired level for each country (e.g. cold For Russia and heat for Egypt). Furthermore, if geoengineering is used as a substitute for emission reductions the acidification of oceans will be intensified. The most serious drawbacks of geoengineering are that we don't really know the consequences on the earth's climate and that once the scientists apply these methods there is no way back [5].

In this paper, we try to combine the traditional approach to policy design of mitigation ,which is the limitation of CO_2 emissions and other $GHGs$ in the atmosphere, with the most recent approach of geoengineering. The model we develop consists of an economic module which includes variables such as output, capital stock, consumption, costs and so on, along a climate module which includes variables such as incoming radiation, albedo, measure of climate's sensitivity, accumulation of $GHGs$ and so on. The main purpose is to study optimal policy design in terms of mitigation and geoengineering efforts in an international or global environment. Our aim is to characterize and contrast noncooperative and cooperative pollution strategies. We consider a world made of N identical countries engaged in production activities that generate pollution which accumulates over time. In the first scenario there is international coordination for the implementation of geoengineering and mitigation in order to maximize their joint payoff. In the second scenario, each government chooses the implementation of geoengineering and mitigation without any consideration of the adverse effects on the other countries. In this scenario, we analyze open-loop and feedback Nash equilibrium strategies. We are interested in seeing whether the two policy instruments can be substitutes or complements and to define the impact of geoengineering costs on the optimal policy mix.

2 The Basic Model

2.1 A General Description of the Model

We start by describing first the economic module. The model economy is characterized by the following assumptions :

$$Y = F(K, \mathcal{A}N) \ , \mathcal{A} = \mathcal{A}_0 e^{\pi t} \ , N = N_0 e^{nt} \tag{1}$$

$$\dot{K} = (1 - X - \vartheta Z) Y(t) - C - \delta K \tag{2}$$

where $t \in [0, \infty)$ is the time index. Expression (1) is technology which assumes that aggregate output, Y, is produced by means of capital, K, and efficient labor, $\mathcal{A}N$, according to a linearly homogeneous production function $F(K, \mathcal{A}N)$ displaying positive and strictly decreasing marginal productivity in both factors and satisfying the Inada conditions. Population N grows at the exogenous rate $n > 0$, and labor efficiency $\mathcal{A}$ grows at the given rate of labor-augmenting technical progress $\pi > 0$. Expression (2) is the accumulation constraint,where C, is the sum of consumption, X, are mitigation expenditures, Z, is geoengineering, ϑ, is the weight on lost output from geoengineering and δ is the rate of physical depreciation of capital .

We normalize all the relevant variables in terms of labor efficiency units. Effective labour is $\mathcal{A}N = \mathcal{A}_0 N_0 e^{(n+\pi)t}$, so we have that output per effective worker is :

$$Y = F(K, \mathcal{A}N) \rightarrow \frac{F(K, \mathcal{A}N)}{\mathcal{A}N} = \frac{Y}{\mathcal{A}N} = y$$

$$f(k) = F\left(\frac{K}{\mathcal{A}N}, \frac{\mathcal{A}N}{\mathcal{A}N}\right) = F(k, 1)$$

We assume that output has a Cobb-Douglas form , so we have that :

$$y = f(k) = k^{\alpha} \quad , \quad 0 < \alpha < 1 \tag{3}$$

Now we can define geoengineering expenditure as proportion of GDP as $\zeta = \frac{Z}{Y}$ and mitigation expenditures as proportion of GDP as $\chi = \frac{X}{Y}$

We set $y \equiv \frac{Y}{\mathcal{A}N}$, $k \equiv \frac{K}{\mathcal{A}N}$, $c = \frac{C}{\mathcal{A}N}$

So the capital stock accumulation equation can be written as :

$$\dot{k} = (1 - \chi - \vartheta\zeta) k^{\alpha} - c - (n + \pi + \delta) k \tag{4}$$

where $f(k)$: output per effective worker , $y = f(k) = k^{\alpha}$: production function , δ : depreciation rate , n : growth rate of population , π : growth rate of labor efficiency (given the rate of labor-augmenting technical progress) , c : consumption per capita.

We assume the utility function :

$$U(C) = C, c = \frac{C}{\mathcal{A}N}, \ C = \mathcal{A}Nc = \mathcal{A}_0 N_0 e^{(n+\pi)t} c \tag{5}$$

where $U(C)$: instantaneous utility function , $U' > 0, U'' < 0, U'(0) = \infty, U'(\infty) = 0$, $\nu = \rho - n - \pi > 0$: social discount rate , N : population, which grows at the exogenous rate $n > 0$ and we have that $N(t) = N_0 e^{nt}$.

For country $i = 1, ...\mathcal{N}$

$$U(C_i) = \mathcal{A}_{0i} N_{0i} e^{(n+\pi)t} c_i \tag{6}$$

due to the linear utility function the problem can be written as a Most Rapid Approach Path (MRAP) problem.

As noted by Weitzman [27] we have two damage functions related to utility. The first one is the damage function from the increase of temperature

$$\Omega_T(T) = \frac{1}{2} c_T T^2, \Omega_T(0) = 0, (\Omega_T(T))' > 0, (\Omega_T(T))'' > 0 \tag{7}$$

The second is the damage function from geoengineering,such as ocean acidification. The use of geoengineering methods from other countries could cause ocean acidification,although the natural absorption of CO_2 by the world's oceans helps mitigate the climatic effects of anthropogenic emissions of CO_2, it is believed that the resulting decrease in pH will have negative consequences, primarily for oceanic calcifying organisms [5]. The impact will be visible to the marine environment of country i . Assume that a country undertakes geoengineering ζ_i,the damages from this will be spread among all countries according to the damage function $\frac{1}{2\mathcal{N}} c_\zeta \zeta_i^2$. If all countries undertake geoengineering total damages will be :

$$\Omega_\zeta(\zeta) = \sum_{i=1}^{\mathcal{N}} \frac{1}{2\mathcal{N}} c_\zeta \zeta_i^2 = \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 \tag{8}$$

$$\Omega_\zeta(0) = 0, (\Omega_\zeta(\zeta))' > 0, (\Omega_\zeta(\zeta))'' > 0$$

There is a subtle difference between damages from the overall increase in temperature due to global warming and the damages from acidification due to geoengineering. Damages from an overall increase in temperature have public bad characteristics since all countries are affected by the global change. On the other hand ocean acidification or acid depositions due to geoengineering will have local effects depending on countries' reliance to oceans, soil compositions etc. Thus damages induced from a given geoengineering activity undertaken by a single country will spread among countries according to countries' specific characteristics. In an extreme case geoengineering activities undertaken by one country might have negligible effects on this country. In our approach we assume that geoengineering damages are evenly spread across country. This a strong simplifying assumption and analysis of asymmetric damages is an interesting area for further research.

The continuous time welfare to be maximized is discounted life time utility of representative consumer minus the damages related to the increase in global

temperature and to geoengineering. The planner solves the MRAP problem :

$$W = \max_{k,\zeta,\chi} \int_0^{\infty} e^{-\nu t} \left\{ \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} c_i - \sum_{i=1}^{\mathcal{N}} \frac{1}{2} c_T T^2 - \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 \right\} dt,\ v = \rho - n - \pi \tag{9}$$

Second we describe the climate module which will be coupled with the economic module. The infrared radiation to space I can be represented as a linear function of the surface temperature T (in degrees Celsius) as in North $[1-3]$:

$$I = A + BT \tag{10}$$

$A = 201.4 W/m^2,\ B = 1.45 W/m^2$, T is the average global surface temperature $\left({}^0C\right)$

We notice that the outgoing radiation may be determined by prescribing the surface temperature.

The equation of the average global temperature T can be written as :

$$\dot{T} = T_0 - T_1 \sum_{i=1}^{\mathcal{N}} \zeta_i + T_2 \left(G - G_0\right) \tag{11}$$

$$where \quad T_0 = \frac{-\left(A + BT\right) + S\left(1-a\right)}{B}, T_1 = \frac{\phi}{B}, T_2 = \frac{\psi}{B} \tag{12}$$

so we have that :

$$\dot{T} = \frac{-\left(A + BT\right) + S\left(1-a\right)}{B} - \frac{\phi}{B} \sum_{i=1}^{\mathcal{N}} \zeta_i + \frac{\psi}{B} \left(G - G_0\right) \tag{13}$$

Variable S is the incoming radiation and a is the average albedo of the planet.

Function $\varphi\left(\zeta\right) = \frac{\phi}{B} \sum_{i=1}^{\mathcal{N}} \zeta_i$ is the reduction in solar radiation due to geoengineering ζ , $\varphi\left(0\right) = 0, \varphi' > 0, \varphi'' < 0$. Where ϕ is the weight of geoengineering at the reduction of the average global temperature.

Variable ψ is a measure of climate's sensitivity and variables G, G_0 are variables which are associated to the greenhouse gasses,where G is the current accumulation of GHGs and G_0 is the preindustrial GHGs accumulation.

So from the previous variables we have that : a) the average global temperature increases when current accumulation of GHGs are above the preindustrial GHGs accumulation because greenhouse gasses block outgoing radiation and b) the average global temperature decreases when the application of geoengineering manage to reduce incoming radiation.

We assume that capital k and average global temperature T converge faster to their corresponding steady states than the variable of the accumulation of GHGs $\left(G\right)$. Then the steady state value of T can be used to express T as a function of G as

$$\dot{T} = 0 \Longrightarrow \frac{-\left(A + BT\right) + S\left(1-a\right)}{B} - \frac{\phi}{B} \left(\zeta_i + \sum_{j \neq i}^{\mathcal{N}} \bar{\zeta}_j \right) + \frac{\psi}{B} \left(G - G_0\right) = 0$$

$$T = \frac{-A + S(1-a) - \phi\left(\zeta_i + \sum_{j\neq i}^{\mathcal{N}} \bar{\zeta}_j\right) + \psi(G - G_0)}{B} = \varphi(\zeta_i, G) \qquad (14)$$

While the fast convergence of k allows to express c as a function of k

$$\dot{k}_i = 0 \Longrightarrow (1 - \chi_i - \vartheta\zeta_i) k_i^{\alpha} - c_i - (\rho + n + \pi + \delta) k_i = 0$$

$$c_i = (1 - \chi_i - \vartheta\zeta_i) k_i^{\alpha} - (n + \pi + \delta) k_i = \gamma(\chi_i, \zeta_i, k_i) \qquad (15)$$

The equation of the accumulation of GHGs can be written as :

$$\begin{aligned}
\dot{G} &= \beta E - mG \\
E &= \sum_{i=1}^{\mathcal{N}} \Omega(t)(1-\chi_i)^{\varepsilon} F(K, \mathcal{A}N) = \sum_{i=1}^{\mathcal{N}} \Omega(t)(1-\chi_i)^{\varepsilon} f(k) \mathcal{A}N \qquad (16) \\
\Omega(t) &= \omega e^{-(n+\pi)t} \\
E &= \sum_{i=1}^{\mathcal{N}} \omega e^{-(n+\pi)t}(1-\chi_i)^{\varepsilon} f(k) \mathcal{A}N = \sum_{i=1}^{\mathcal{N}} \omega(1-\chi_i)^{\varepsilon} f(k)
\end{aligned}$$

$$\dot{G} = \sum_{i=1}^{\mathcal{N}} \beta\omega(1-\chi_i)^{\varepsilon} k_i^{\alpha} - mG \qquad (17)$$

where χ are mitigation expenditures as proportion of GDP and variable m is natural decay rate of GHGs.

In the approach by Criado, Valente, Stengos [22], the product of $\Omega(t)(1-\chi)^{\varepsilon}$ is aggregate emission intensity with $\Omega(t)$ to be the baseline emission intensity and $(1-\chi)^{\varepsilon}$ is a function representing the effects of mitigation, $\varepsilon > 1$. From (16) ,maximal mitigation effort, $\chi = f(k)$, implies zero emissions, whereas zero mitigation expenditures, $\chi = 0$, imply that the emission intensity equals the baseline level. Also the baseline intensity varies over time as it is influenced by technological progress. Explosive dynamics in emissions per capita can be avoided only if the rate of emission-reducing progress is at least equal to the rate of output-augmenting technical progress. In the present model,this sustainability condition corresponds to $\dot{\Omega}(t)/\Omega(t) \leq -\pi$. In order to ensure that stabilizing per capita emissions in the longrun is technically feasible,we assume symmetric rates of emission-reducing and labor-augmenting progress by setting $\Omega(t) = \omega e^{-(n+\pi)t}$.

The problem of the social planner is to maximize discounted life time utility by choosing paths for mitigation expenditures χ (control of emissions of CO2 and other GHGs) ,geoengineering $\zeta(t)$ and consumption $c(t)$ subject to to the capital stock accumulation equation , the equation of the average global temperature and the equation of the accumulation of GHGs in order to have optimal policy design that takes into account both mitigation and geoengineering (increase of albedo) and to find a critical path which will provide a stable solution for managing global warming in an optimal way.

3 The Model with Cooperation among countries

We assume that $i = 1, ..., \mathcal{N}$ countries are involved in emissions and geoengineering.

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i,\zeta_i,\chi_i} \int_0^\infty e^{-\nu t} \left[\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} c_i - \frac{1}{2} \sum_{i=1}^{\mathcal{N}} c_T T^2 - \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 \right] dt \quad (18) \\ v &= \rho - n - \pi \end{aligned}$$

subject to

$$\dot{G} = \sum_{i=1}^{\mathcal{N}} \beta\omega \left(1 - \chi_i\right)^\varepsilon k_i^\alpha - mG \quad (19)$$

Using the most rapid approach path transformation the optimization problem can be written as:

$$\begin{aligned} W_i &= \max_{k,\zeta,\chi} \int_0^\infty e^{-\nu t} \left\{ \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \left[\left(1 - \chi_i - \vartheta\zeta_i\right) k_i^\alpha - \left(\rho + \delta\right) k_i \right] \right. \\ &\quad \left. - \frac{1}{2} \sum_{i=1}^{\mathcal{N}} c_T T^2 - \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 \right\} dt \end{aligned}$$

The generalized current value Hamiltonian function :

$$\begin{aligned} \Longrightarrow \quad H &= \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \gamma\left(\chi_i, \zeta_i, k_i\right) - \frac{1}{2} c_T \sum_{i=1}^{\mathcal{N}} \left(\varphi\left(\zeta_i, G\right)\right)^2 - \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 + \\ &\quad + \lambda_G\left(t\right) \left[\sum_{i=1}^{\mathcal{N}} \beta\omega \left(1 - \chi_i\right)^\varepsilon k_i^\alpha - mG \right] \quad (20) \end{aligned}$$

This implies that the optimal controls will be some functions of the (G, λ_G) or[1]

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G \beta\omega \left(1 - \chi_i\right)^\varepsilon + \mathcal{A}_{0i} N_{0i} \left(1 - \chi_i - \vartheta\zeta_i\right)\right)}{\mathcal{A}_{0i} N_{0i} \left(\rho + \delta\right)} \right)^{\frac{1}{1-\alpha}} = h_1\left(G, \lambda_G\right) \quad (21) \\ \chi^* &= 1 - \left(-\frac{\mathcal{A}_{0i} N_{0i}}{\varepsilon \lambda_G \beta\omega} \right)^{\frac{1}{\varepsilon - 1}} = h_2\left(\lambda_G\right) \quad (22) \\ \zeta^* &= \frac{\mathcal{N}^2 \phi c_T \left[-A + S\left(1 - a\right) + \psi\left(G - G_0\right)\right] - \mathcal{N} B^2 \mathcal{A}_{0i} N_{0i} \vartheta k_i^a}{c_\zeta B^2 + \mathcal{N}^3 \phi^2 c_T} \quad (23) \\ &= h_3\left(G, \lambda_G\right) \end{aligned}$$

[1] For derivation of (21), (22), (23) see Appendix

we notice that at equation (21) the optimal level of capital accumulation at time t, is determined by the equality of the extra benefits, in terms of consumption, from having an additional unit of capital with the extra cost of the global damages due to the GHGs generated by this unit. At equation (22) the optimal level of mitigation expenditures is determined at the point where the extra global benefits from an additional unit of mitigation, in terms of reduced GHGs, equals the cost of mitigation in terms of lost output. At equation (23) the optimal level of geoengineering is determined by the equality of the extra global benefits from an additional unit of geoengineering, in terms of reduced solar radiation, with the cost of geoengineering in terms of lost output.

The Modified Hamiltonian System characterizing Cooperation is [2]:

$$\dot{G} = \sum_{i=1}^{\mathcal{N}} \beta\omega \left(1 - h_2\left(\lambda_G\right)\right)^{\varepsilon} \left[h_1\left(G, \lambda_G\right)\right]^{\alpha} - mG \tag{24}$$

$$\dot{\lambda}_G^c = \left(m + \nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T T}{B} \tag{25}$$

$$\text{where } T = \frac{-A + S\left(1 - a\right) - \phi \sum_{i=1}^{\mathcal{N}} \zeta^*\left(G, \lambda_G\right) + \psi\left(G - G_0\right)}{B}$$

Intertemporal transversality conditions :

$$\lim_{T \longrightarrow \infty} e^{-\nu t} \lambda_G G\left(t\right) dt = 0 \ , \quad \lim_{T \longrightarrow \infty} e^{-\nu t} G\left(t\right) dt = 0 \tag{26}$$

The modified Hamiltonian dynamic system of (24) , (25) with initial condition for G and the transversality condition (26) determine the evolution of the state (G^*) and costate variables (λ_G) along the optimal path which characterizes optimal mitigation expenditures and optimal geoengeneering.

4 The Model with Non-Cooperation among countries (Open Loop Nash equilibrium)

In this model we have a country i which is not cooperate with country j for the implementation of geoengineering and mitigation. Thus country i takes the action of country j as given in an open loop Nash equilibrium.

The noncooperative solution for the country i is

$$W = \max_{k,\zeta,\chi} \int_0^{\infty} e^{-\nu t} \left[\mathcal{A}_{0i} N_{0i} c_i - \frac{1}{2} c_T T^2 - \frac{c_\zeta}{2\mathcal{N}} \left(\zeta_i^2 + \sum_{j \neq i}^{\mathcal{N}} \bar{\zeta}_j^2\right)\right] dt \tag{27}$$

$$v = \rho - n - \pi$$

[2] For derivation of (24) , (25) see Appendix

subject to

$$\dot{G} = \beta\omega \left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} + \sum_{j\neq i}^{\mathcal{N}} \beta\omega \left(1-\bar{\chi}_j\right)^{\varepsilon} \bar{k}_j^{\alpha} - mG \tag{28}$$

where $\bar{\zeta}_j$ means that country i takes the action of country j as given in an open loop Nash equilibrium.

The generalized current value noncooperative Hamiltonian function :

$$\begin{aligned} H^{OL} &= \mathcal{A}_{0i} N_{0i} \gamma\left(\chi_i, \zeta_i, k_i\right) - \frac{1}{2} c_T \left(\varphi\left(\zeta_i, G\right)\right)^2 - \frac{c_\zeta}{2\mathcal{N}} \left(\zeta_i^2 + \sum_{j\neq i}^{\mathcal{N}} \bar{\zeta}_j^2\right) + \\ &\quad + \lambda_G(t) \left[\beta\omega \left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} + \sum_{j\neq i}^{\mathcal{N}} \beta\omega \left(1-\bar{\chi}_j\right)^{\varepsilon} \bar{k}_j^{\alpha} - mG\right] \end{aligned} \tag{29}$$

we impose symmetry so $\zeta_i = \bar{\zeta}_j = \zeta$ and $k_i = k$ and $\chi_i = \chi$, $\mathcal{A}_{0i} = \mathcal{A}_0, N_{0i} = N_0$ for all i.

This implies that the optimal controls will be some functions of the (G, λ_G) or[3]

$$k^* = \left(\frac{a\left(\lambda_G \beta\omega \left(1-\chi\right)^{\varepsilon} + \mathcal{A}_0 N_0 \left(1-\chi-\vartheta\zeta\right)\right)}{\mathcal{A}_0 N_0 \left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}} = h_1\left(G, \lambda_G\right) \tag{30}$$

$$\chi^* = 1 - \left(-\frac{\mathcal{A}_0 N_0}{\varepsilon \lambda_G \beta\omega}\right)^{\frac{1}{\varepsilon-1}} = h_2\left(\lambda_G\right) \tag{31}$$

$$\begin{aligned} \zeta^* &= \frac{\mathcal{N}\phi c_T \left[-A + S\left(1-a\right) + \psi\left(G-G_0\right)\right] - \mathcal{N}B^2 \mathcal{A}_0 N_0 \vartheta k^a}{c_\zeta B^2 + \mathcal{N}^2 \phi^2 c_T} \\ &= h_3\left(G, \lambda_G\right) \end{aligned} \tag{32}$$

The interpretation of the optimally conditions is the same as in cooperative equilibrium. But the benefits or the costs in terms of changes in the GHGs accumulation are not global but they refer to a single country.

The Modified Hamiltonian System characterizing the Open Loop Nash equilibrium under symmetry implies[4]

$$\dot{G} = \mathcal{N}\beta\omega \left[1 - h_2\left(\lambda_G\right)\right]^{\varepsilon} \left[h_1\left(G, \lambda_G\right)\right]^{\alpha} - mG \tag{33}$$

$$\dot{\lambda}_G = \left(m+\nu\right)\lambda_G + \frac{\psi c_T T}{B} \tag{34}$$

$$\text{where } T = \frac{-A + S\left(1-a\right) + \psi\left(G-G_0\right) - \mathcal{N}\phi\zeta^*\left(G, \lambda_G\right)}{B}$$

[3] For derivation of (30), (31), (32) see Appendix

[4] For derivation of (33), (34) see Appendix

Intertemporal transversality conditions :

$$\lim_{T\longrightarrow\infty} e^{-\nu t}\lambda_G G(t)\,dt = 0\ ,\quad \lim_{T\longrightarrow\infty} e^{-\nu t} G(t)\,dt = 0 \tag{35}$$

The modified Hamiltonian dynamic system of (33), (34) with initial condition for G and the transversality condition (35) determine the evolution of the state (G^*) and costate variables (λ_G) along the optimal path which characterizes optimal mitigation expenditures and optimal geoengeneering.

5 The Model with Non-Cooperation among countries (Feedback Nash equilibrium)

In this model we have a country i which does not cooperate with any other country $j \neq i$ for the implementation of geoengineering and mitigation. Country i takes as given the feedback or closed loop time stationary strategies $\zeta_j(G(t))$ and $\chi_j(G(t))$ $j \neq i$, of other countries. The feedback strategies condition mitigation and geoengineering policies on the observed concentration of CO_2. Regarding the stock of capital we don't keep the assumption of open-loop strategies since it seems not realistic, at least at the current world, to assume that aggregate investment in a country is conditioned on CO_2 accumulation.[5]

The noncooperative feedback solution for the country i is determined by the solution of the problem

$$\begin{aligned} W_i &= \max_{k,\zeta,\chi} \int_0^\infty e^{-\nu t}\left[\mathcal{A}_{0i}N_{0i}c_i - \frac{1}{2}c_T T^2 - \frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2 + \sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j(G)\right)^2\right)\right] dt \\ v &= \rho - n - \pi > 0 \end{aligned} \tag{36}$$

subject to

$$\dot{G} = \beta\omega\left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} + \sum_{j\neq i}^{\mathcal{N}} \beta\omega\left(1-\chi_j(G)\right)^{\varepsilon}\bar{k}_j^{\alpha} - mG \tag{37}$$

Using the most rapid approach path transformation the optimization problem can be written as:

$$\begin{aligned} W_i = \max_{k,\zeta,\chi} \int_0^\infty e^{-\nu t}\,[&\mathcal{A}_{0i}N_{0i}\left(1-\chi_i-\vartheta\zeta_i\right)k_i^{\alpha}c_i - (\rho+\delta)k_i \\ &-\frac{1}{2}c_T T^2 - \frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2 + \sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j(G)\right)^2\right)\Bigg]\,dt \end{aligned}$$

[5] One might argue that investment in green technologies could be conditioned on CO_2 accumulation. This requires however a disaggregated model of technology choice which is beyond the scope of the present paper.

The generalized current value noncooperative Hamiltonian function :

$$H^F = \mathcal{A}_{0i}N_{0i}\gamma\left(\chi_i,\zeta_i,k_i\right)-\frac{1}{2}c_T\left(\varphi\left(\zeta_i,G\right)\right)^2-\frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)^2\right)+$$
$$+\lambda_G\left(t\right)\left[\beta\omega\left(1-\chi_i\right)^\varepsilon k_i^\alpha+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\left(1-\chi_j\left(G\right)\right)^\varepsilon\bar{k}_j^\alpha-mG\right] \quad (38)$$

This implies that the optimal controls will be some functions of the (G,λ_G) or[6]

$$k^* = \left(\frac{a\left(\lambda_G\beta\omega\left(1-\chi_i\right)^\varepsilon+\mathcal{A}_{0i}N_{0i}\left(1-\chi_i-\vartheta\zeta_i\right)\right)}{\mathcal{A}_{0i}N_{0i}\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}}=h_1\left(G,\lambda_G\right) \quad (39)$$

$$\chi^* = 1-\left(-\frac{\mathcal{A}_{0i}N_{0i}}{\varepsilon\lambda_G\beta\omega}\right)^{\frac{1}{\varepsilon-1}}=h_2\left(\lambda_G\right) \quad (40)$$

$$\zeta^* = \frac{\mathcal{N}\phi c_T\left[-A+S\left(1-a\right)-\phi\sum_{j\neq i}^{\mathcal{N}}\zeta_j\left(G\right)+\psi\left(G-G_0\right)\right]-\mathcal{N}B^2\mathcal{A}_{0i}N_{0i}\vartheta k_i^a}{c_\zeta B^2+\mathcal{N}\phi^2c_T} \quad (41)$$
$$= h_3\left(G\right)$$

The interpretation of the optimally conditions is the same as in cooperative equilibrium. But the benefits or the costs in terms of changes in the GHGs accumulation are not global but they refer to a single country i, as i takes as given the feedback or closed loop time stationary strategies $\zeta_j\left(G\left(t\right)\right)$ and $\chi_j\left(G\left(t\right)\right)$ $j\neq i$, of other countries.

The Modified Hamiltonian System characterizing the feedback Nash equilibrium (FBNE) under symmetry implies[7]

$$\dot{G}=\mathcal{N}\beta\omega\left(1-h_2\left(\lambda_G\right)\right)^\varepsilon\left[h_1\left(G,\lambda_G\right)\right]^a-mG \quad (42)$$

$$\dot{\lambda}_G^F = \left(m+\nu\right)\lambda_G+c_TT\left[\frac{\left(-\right)\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+c_\zeta\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right)$$
$$+\varepsilon\lambda_G\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^\alpha\left(\chi_j\left(G\right)\right)' \quad (43)$$

$$\text{where } T=\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)\right)+\psi\left(G-G_0\right)}{B}$$

Intertemporal transversality conditions :

$$\lim_{T\longrightarrow\infty}e^{-\nu t}\lambda_GG\left(t\right)dt=0\ ,\quad \lim_{T\longrightarrow\infty}e^{-\nu t}G\left(t\right)dt=0 \quad (44)$$

[6] For derivation of (39), (40), (41) see Appendix
[7] For derivation of (42), (43) see Appendix

The modified Hamiltonian dynamic system of (42) , (43) with initial condition for G and the transversality condition (44) determine the evolution of the state (G^*) and costate variables (λ_G) along the optimal path which characterizes optimal mitigation expenditures and optimal geoengeneering.

6 Comparison

In this section we compare the three cases (cooperation,open loop,feedback)

Cooperative

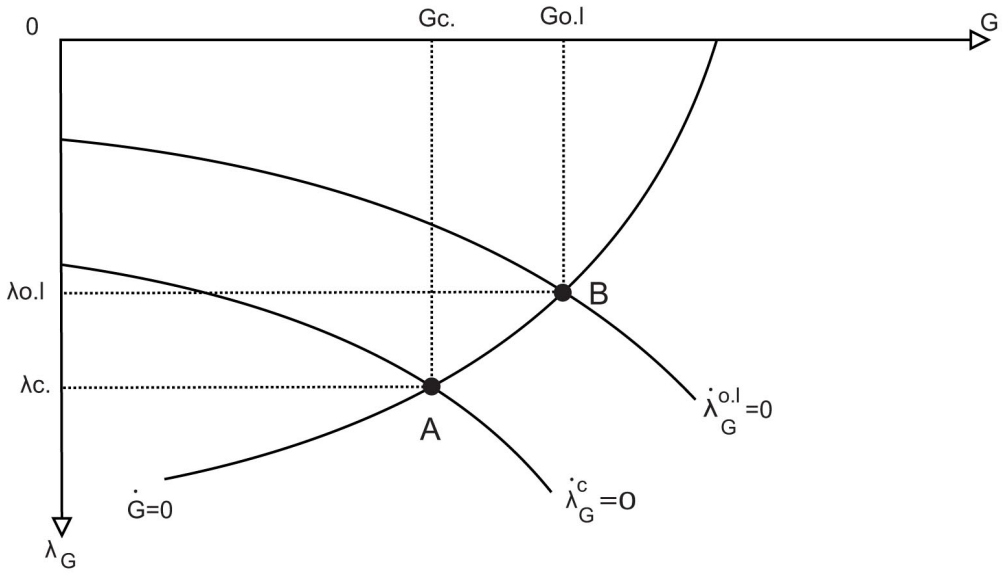
$$\lambda_G^C = \frac{-c_T \mathcal{N} T \frac{\psi}{B}}{(m+\nu)} \tag{45}$$

Open loop

$$\lambda_G^{O.L} = \frac{-c_T T \frac{\psi}{B}}{(m+\nu)} \tag{46}$$

By comparing $\dot{\lambda}_G^{O.L} = 0$ and $\dot{\lambda}_G^C = 0$ for any given G we can see that :

$$\left|\lambda_G^C\right| = \mathcal{N}\left|\lambda_G^{O.L}\right| \Longrightarrow \left|\lambda_G^{O.L}\right| < \left|\lambda_G^C\right| \tag{47}$$



We have that in absolute values the shadow cost of GHGs accumulation in the case of cooperation among countries is higher than the shadow cost in Open Loop Nash Equilibrium. If we relax the constraint of the level of current accumulation of GHGs and allow a higher level then the cost from this decision

will be greater at a cooperative equilibrium than at an OLNE. This is because in an OLNE countries commit themselves to a particular discharge path at the outset of the game and do not respond to observed variations of the GHGs concentration. Therefore we have higher Steady State stock of GHGs in OLNE than in the case of cooperation among countries.

Feedback

$$\lambda_G^F = \frac{-c_T T\left[\frac{(-)\phi(\mathcal{N}-1)}{B}\zeta'(G) + \frac{\psi}{B}\right] - \frac{c_\zeta(\mathcal{N}-1)}{\mathcal{N}}\zeta(G)\zeta'(G)}{\left(m+\nu+\varepsilon(\mathcal{N}-1)\beta\omega\left(1-\chi_j(G)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\chi'(G)\right)} \tag{48}$$

Open loop is a special case of the feedback solution for $\zeta'(G)=0$ and $\chi'(G)=0$

$$\begin{aligned}\lambda_G^F &= \frac{-c_T T\left[\frac{(-)\phi(\mathcal{N}-1)}{B}\cdot 0 + \frac{\psi}{B}\right] - \frac{c_\zeta(\mathcal{N}-1)}{\mathcal{N}}\zeta(G)\cdot 0}{\left(m+\nu+\varepsilon(\mathcal{N}-1)\beta\omega\left(1-\chi_j(G)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\cdot 0\right)} \\ \Longrightarrow \quad \lambda_G^F &= \frac{-c_T T\frac{\psi}{B}}{(m+\nu)} = \lambda_G^{O.L}\end{aligned}$$

we need to compare :

$$\lambda_G^{O.L} = \frac{-c_T T\frac{\psi}{B}}{(m+\nu)}$$

$$\lambda_G^F = \frac{-c_T T\frac{\psi}{B} + (\mathcal{N}-1)\zeta'(G)\left[c_T T\frac{\phi}{B} - \frac{c_\zeta}{\mathcal{N}}\zeta(G)\right]}{\left(m+\nu+\varepsilon(\mathcal{N}-1)\beta\omega\left(1-\chi_j(G)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\chi'(G)\right)}$$

To simplify the exposition we assume linear feedback strategies ,so we have that :

$$\begin{aligned}\zeta'(G) &= \xi\ ,\ \xi>0 \\ \chi'(G) &= \mu\ ,\ \mu>0\end{aligned}$$

$$\lambda_G^F = \frac{-c_T T\frac{\psi}{B} + (\mathcal{N}-1)\xi\ \left[c_T T\frac{\phi}{B} - \frac{c_\zeta}{\mathcal{N}}\zeta(G)\right]}{\left(m+\nu+\varepsilon(\mathcal{N}-1)\beta\omega\left(1-\chi_j(G)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\mu\right)}$$

$$\mathcal{N}>1 \rightarrow (\mathcal{N}-1)>0$$

$$\chi'(G)>0\ ,\ \zeta'(G)>0$$

$$\left(m+\nu+\varepsilon(\mathcal{N}-1)\beta\omega\left(1-\chi_j(G)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\mu\right) > (m+\nu)$$

$$we\ set:\ \Gamma = \left[c_T T\frac{\phi}{B} - \frac{c_\zeta}{\mathcal{N}}\zeta(G)\right]$$

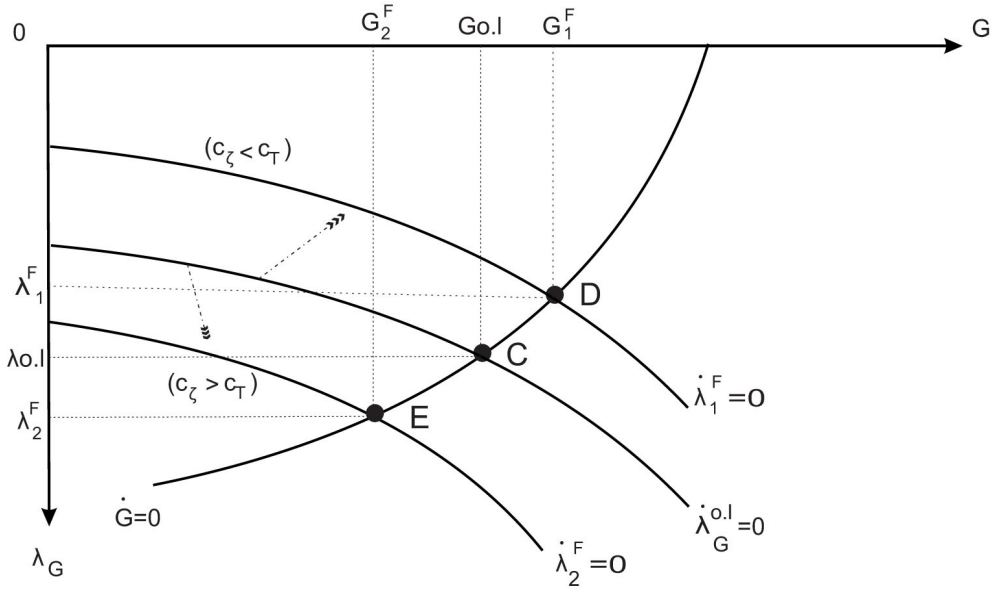
By comparing $\dot{\lambda}_G^{O.L}=0$ and $\dot{\lambda}_G^{F}=0$ for any given G we can see that :

· If the damage cost of geoengineering is low ($c_\zeta < c_T$) then $\Longrightarrow$ $\Gamma > 0$ then the nominator is positive and we have that :

$$\left|-c_T T\frac{\psi}{B}\right| > \left|-c_T T\frac{\psi}{B}+\Gamma\right| \text{ then } \left|\lambda_G^{O.L}\right| > \left|\lambda_G^F\right|$$

· If the damage cost of geoengineering is high ($c_\zeta > c_T$) then $\Longrightarrow$ $\Gamma < 0$ then the nominator is negative and we have that :

$$\text{if } \left|-c_T T\frac{\psi}{B}\right| < \left|-c_T T\frac{\psi}{B}+\Gamma\right| \text{ then } \left|\lambda_G^{O.L}\right| < \left|\lambda_G^F\right|$$



Proposition

The steady state shadow cost of GHGs accumulation at an open loop Nash equilibrium common to all firms $\left|\lambda_G^{O.L}\right| = \left|\frac{-c_T T\frac{\psi}{B}}{(m+\nu)}\right|$ compares as follows to the social optimum and to the feedback Nash level :

$$\begin{aligned}
&i) && \left|\lambda_G^{O.L}\right| && < && |\lambda_G^c| && \\
&ii) && \left|\lambda_G^{O.L}\right| && > && \left|\lambda_G^F\right| && \text{If the cost of geoeng. is low } (c_\zeta < c_T)\\
&iii) && \left|\lambda_G^{O.L}\right| && < && \left|\lambda_G^F\right| && \text{If the cost of geoeng. is high } (c_\zeta > c_T)
\end{aligned}$$

at the first case we have that in absolute values the shadow cost of GHGs accumulation in the case of cooperation among countries is higher than the shadow cost in Open Loop Nash Equilibrium.

at the second and third case we have that in absolute values shadow cost of GHGs accumulation in Open Loop Nash Equilibrium can be higher or lower than the shadow cost in Feedback Nash Equilibrium depending on the level of the damage cost of geoengineering (c_ζ). The traditional result is obtained when the damage cost of geoengineering is lower than the damage cost from the increase of temperature (c_T) . Then the effect of free riding in the FBNE is very strong and the shadow cost of GHGs accumulation in OLNE is greater than the one in FBNE. If the damage cost of geoengineering is higher than the damage cost from the increase of temperature then this cost can compensate the effect of free riding in the FBNE and the shadow cost of GHGs accumulation in FBNE is greater than the one in OLNE.

Existing literature suggests that shadow cost of GHGs concentration will be higher in a OLNE than in a FBNE due to the free riding effect. We showed that if the damage cost of geoengineering is too high,then it is possible to compensate the effect of free riding and the shadow cost of OLNE can be lower than the shadow cost of FBNE.

7 Stability

$$\dot{G} = \mathcal{N}\beta\omega\left(1-h_2\left(\lambda_G\right)\right)^{\varepsilon}\left[h_1\left(G,\lambda_G\right)\right]^{\alpha} - mG = 0 \tag{49}$$

$$\dot{\lambda}_G^{c} = \left(m+\nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T}{B}\cdot\frac{-A+S\left(1-a\right)-\phi\mathcal{N}h_3\left(G,\lambda_G\right)+\psi\left(G-G_0\right)}{B} = 0 \tag{50}$$

$$\dot{\lambda}_G^{O.L} = \left(m+\nu\right)\lambda_G + \frac{\psi c_T}{B}\cdot\frac{-A+S\left(1-a\right)-\phi\mathcal{N}h_3\left(G,\lambda_G\right)+\psi\left(G-G_0\right)}{B} = 0 \tag{51}$$

$$\begin{aligned}\dot{\lambda}_G^{F} &= \left(m+\nu+\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\left(\chi_j\left(G\right)\right)'\right)\lambda_G \\ &+c_T\frac{-A+S\left(1-a\right)-\phi\mathcal{N}\zeta\left(G\right)+\psi\left(G-G_0\right)}{B} \\ &\cdot\left[-\frac{\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+c_\zeta\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right)=0\end{aligned} \tag{52}$$

if we have one intersection (at a low or at a high level of current accumulation of GHGs) between $\dot{G}=0$ and $\dot{\lambda}_G=0$ then we will have local Saddle point[8] (negative Jacobian determinant) around the equilibrium point

$$G^* = \frac{1}{m}\left(\mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G,\lambda_G\right)\right]^{\alpha}\right)$$

The Steady State stock of GHGs accumulation under cooperation will be lower than the Steady State stock of GHGs accumulation under OLNE,due to the first case of Proposition ($|\lambda_G^o| < |\lambda_G^c|$) .

[8] For proof, see Appendix

If the damage cost of geoengineering (c_ζ) is lower than the damage cost from the increase of temperature (c_T) then Steady State stock of GHGs accumulation in OLNE is lower than the one in FBNE ,due to the second case of Proposition $\left(|\lambda_G^o| > \left|\lambda_G^F\right|\right)$.

If the damage cost of geoengineering (c_ζ) is higher than the damage cost from the increase of temperature (c_T) then Steady State stock of GHGs accumulation in OLNE is higher than the one in FBNE ,due to the third case of Proposition $\left(|\lambda_G^o| < \left|\lambda_G^F\right|\right)$.

8 Marginal Rate of Substitution

We can derive the marginal rate of substitution between mitigation expenditures and geoengineering,which is the level of mitigation for which the policy maker is willing to exchange for cost of geoengineering,while keeping his overall utility constant.

$$MRS_{\chi_i,\zeta_i} = \frac{\partial H^*/\partial \chi_i}{\partial H^*/\partial \zeta_i} = \frac{d\zeta_i}{d\chi_i}$$

where H^* is the Hamiltonian along the optimal trajectory. Since the Hamiltonian is constant along the optimal trajectory MRS_{χ_i,ζ_i} can be interpreted as the rate at which the policy maker is willing to exchange mitigation along the optimal trajectory. We can see that if mitigation expenditures increase then the policy maker is more willing to exchange mitigation with geoengineering along the optimal trajectory.

For Cooperation among countries we have :

$$MRS^C_{\chi_i,\zeta_i} = \frac{-\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} k_i^\alpha - \lambda_G \sum_{i=1}^{\mathcal{N}} \beta\omega\varepsilon \left(1-\chi_i\right)^{\varepsilon-1} k_i^\alpha}{-\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \vartheta k_i^\alpha - \sum_{i=1}^{\mathcal{N}} \frac{c_\zeta}{\mathcal{N}} \zeta_i + \sum_{i=1}^{\mathcal{N}} \frac{\mathcal{N}\phi c_T[-A+S(1-a)-\phi\mathcal{N}\zeta_i+\psi(G-G_0)]}{B^2}} \tag{53}$$

For Open Loop Nash Equilibrium we have :

$$MRS^{O.L}_{\chi,\zeta} = \frac{-\mathcal{A}_0 N_0 k^\alpha - \lambda_G \beta\omega\varepsilon \left(1-\chi\right)^{\varepsilon-1} k^\alpha}{-\mathcal{A}_0 N_0 \vartheta k^\alpha - \frac{c_\zeta}{\mathcal{N}}\zeta + \frac{\phi c_T[-A+S(1-a)-\phi\mathcal{N}\zeta+\psi(G-G_0)]}{B^2}} \tag{54}$$

For Feedback Nash Equilibrium we have :

$$MRS^{FB}_{\chi_i,\zeta_i} = \frac{-\mathcal{A}_{0i} N_{0i} k_i^\alpha - \lambda_G \beta\omega\varepsilon \left(1-\chi_i\right)^{\varepsilon-1} k_i^\alpha}{-\mathcal{A}_{0i} N_{0i} \vartheta k_i^\alpha - \frac{c_\zeta}{\mathcal{N}}\zeta_i + \frac{\phi c_T\left[-A+S(1-a)-\phi\zeta_i-\phi\sum_{j\neq i}^{\mathcal{N}}\zeta_j(G)+\psi(G-G_0)\right]}{B^2}} \tag{55}$$

We assume that the level of geoengineering (ζ) is the same for all cases ($\zeta_i^C = \zeta_i^{O.L} = \zeta_i^{FB} = \zeta_j(G)$).

we have that :

$$\begin{aligned} i)\ \text{From}\ (21),(30) &: k_i^C = k_i^{O.L} \\ ii)\ \text{From}\ (22),(31) &: \chi_i^C = \chi_i^{O.L} \\ iii)\ \text{we assume that} &: \zeta_i^C = \zeta_i^{O.L} \end{aligned}$$

$$\Longrightarrow \mid MRS^C \mid < \mid MRS^{O.L} \mid$$

we have that :

$$\begin{aligned} i)\ \text{From}\ (39),(30) &: k_i^{FB} = k_i^{O.L} \\ ii)\ \text{From}\ (40),(31) &: \chi_i^{FB} = \chi_i^{O.L} \\ iii)\ \text{From}\ (41),(32) &: \zeta_i^{FB} = \zeta_i^{O.L} = \zeta_j(G) \end{aligned}$$

$$\Longrightarrow \mid MRS^{O.L} \mid = \mid MRS^{FB} \mid$$

We can see that the policy maker is more willing to exchange mitigation with geoengineering along the optimal trajectory in the cases of Feedback Nash Equilibrium and Open Loop Nash Equilibrium than in the case of Cooperation. In a Feedback Nash Equilibrium if mitigation expenditures increase then the policy maker will change his policy and exchange mitigation with geoengineering along the optimal trajectory, but in Cooperation the policy maker will not allow a high level of GHGs accumulation in order to have more production and so he will not change his policy and exchange mitigation with geoengineering as easily as in the case of OLNE or FBNE. So we have that in non-cooperation the policy maker has higher incentive for geoengineering in relation to mitigation.

9 Two Extreme Cases

9.1 First Case :The Model without mitigation

9.1.1 Cooperation

We assume that $i = 1, ..., \mathcal{N}$ countries are involved in emissions and geongineering.

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i,\zeta_i} \int_0^\infty e^{-\nu t} \left[\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} c_i - \frac{1}{2} \sum_{i=1}^{\mathcal{N}} c_T T^2 - \frac{c_\zeta}{2\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \zeta_i^2 \right] dt \qquad (56) \\ v &= \rho - n - \pi \end{aligned}$$

subject to

$$\dot{G} = \sum_{i=1}^{\mathcal{N}} \beta \omega k_i^\alpha - mG \qquad (57)$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \left[\left(1-\vartheta\zeta_i\right) k_i^{\alpha} - \left(\rho+\delta\right) k_i \right] - \\ & -\frac{1}{2}\sum_{i=1}^{\mathcal{N}} c_T \left(\frac{-A+S\left(1-a\right)-\phi\sum\limits_{i=1}^{\mathcal{N}}\zeta_i+\psi\left(G-G_0\right)}{B} \right)^2 \\ & -\frac{c_\zeta}{2\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\zeta_i^2 + \lambda_G\left(t\right)\left[\sum_{i=1}^{\mathcal{N}}\beta\omega k_i^{\alpha} - mG\right] \end{aligned}$$

This implies that the optimal controls will be some functions of the (G, λ_G) or

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G\beta\omega + \mathcal{A}_{0i}N_{0i}\left(1-\vartheta\zeta_i\right)\right)}{\mathcal{A}_{0i}N_{0i}\left(\rho+\delta\right)} \right)^{\frac{1}{1-\alpha}} = h_1\left(G,\lambda_G\right) \\ \zeta^* &= \frac{\mathcal{N}^2\phi c_T\left[-A+S\left(1-\alpha\right)+\psi\left(G-G_0\right)\right]-\mathcal{N}B^2\mathcal{A}_{0i}N_{0i}\vartheta k_i^a}{c_\zeta B^2+\mathcal{N}^3\phi^2 c_T} = h_2\left(G,\lambda_G\right) \end{aligned}$$

The Modified Hamiltonian System is :

$$\dot{G} = \sum_{i=1}^{\mathcal{N}}\beta\omega\left(k^*\right)^{\alpha} - mG \tag{58}$$

$$\dot{\lambda}_G^{c,w.m} = \left(m+\nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T}{B}\left(\frac{-A+S\left(1-a\right)-\phi\sum\limits_{i=1}^{\mathcal{N}}\zeta^*+\psi\left(G-G_0\right)}{B} \right) \tag{59}$$

9.1.2 Open Loop

We assume that $i = 1, ..., \mathcal{N}$ countries are involved in emissions and geongineering.

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i,\zeta_i}\int_0^{\infty} e^{-\nu t}\left[\mathcal{A}_{0i}N_{0i}c_i - \frac{1}{2}c_T T^2 - \frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2 + \sum_{j\neq i}^{\mathcal{N}}\bar{\zeta}_j^2\right)\right]dt \quad (60) \\ v &= \rho - n - \pi \end{aligned}$$

subject to

$$\dot{G} = \beta\omega k_i^{\alpha} + \sum_{j\neq i}^{\mathcal{N}}\beta\omega\bar{k}_j^{\alpha} - mG$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \mathcal{A}_{0i}N_{0i}\left[\left(1-\vartheta\zeta_i\right)k_i^{\alpha}-\left(\rho+\delta\right)k_i\right]- \\ &\quad -\frac{1}{2}c_T\left(\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\bar{\zeta}_j\right)+\psi\left(G-G_0\right)}{B}\right)^2 \\ &\quad -\frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2+\sum_{j\neq i}^{\mathcal{N}}\bar{\zeta}_j^2\right)+\lambda_G\left(t\right)\left[\beta\omega k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\bar{k}_j^{\alpha}-mG\right] \end{aligned}$$

The optimal controls will be some functions of the (G,λ_G) and with symmetry we have :

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G\beta\omega+\mathcal{A}_0N_0\left(1-\vartheta\zeta\right)\right)}{\mathcal{A}_0N_0\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}}=h_1\left(G,\lambda_G\right) \\ \zeta^* &= \frac{\mathcal{N}\phi c_T\left[-A+S\left(1-a\right)+\psi\left(G-G_0\right)\right]-\mathcal{N}B^2\mathcal{A}_0N_0\vartheta k^a}{c_\zeta B^2+\mathcal{N}^2\phi^2c_T}=h_2\left(G,\lambda_G\right) \end{aligned}$$

The Modified Hamiltonian System is :

$$\dot{G} = \mathcal{N}\beta\omega\left[k^*\left(G,\lambda_G\right)\right]^{\alpha}-mG \tag{61}$$

$$\dot{\lambda}_G^{O.L,w.m} = \left(m+\nu\right)\lambda_G+\frac{\psi c_T\left[-A+S\left(1-a\right)-\phi\mathcal{N}\zeta^*\left(G,\lambda_G\right)+\psi\left(G-G_0\right)\right]}{B^2} \tag{62}$$

9.1.3 Feedback

We assume that $i=1,...,\mathcal{N}$ countries are involved in emissions and geongineering.

The solution is the maximization of global welfare or

$$W = \max_{k_i,\zeta_i}\int_0^{\infty}e^{-\nu t}\left[\mathcal{A}_{0i}N_{0i}c_i-\frac{1}{2}c_TT^2-\frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)^2\right)\right] \tag{63}$$

$$v = \rho-n-\pi$$

subject to

$$\dot{G}=\beta\omega k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\bar{k}_j^{\alpha}-mG$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \mathcal{A}_{0i}N_{0i}\left[\left(1-\vartheta\zeta_i\right)k_i^{\alpha}-\left(\rho+\delta\right)k_i\right]- \\ &\quad -\frac{1}{2}c_T\left(\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)\right)+\psi\left(G-G_0\right)}{B}\right)^2 \\ &\quad -\frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)^2\right)+\lambda_G\left(t\right)\left[\beta\omega k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\bar{k}_j^{\alpha}-mG\right] \end{aligned}$$

The optimal controls will be some functions of the (G,λ_G) and with symmetry we have :

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G\beta\omega+\mathcal{A}_0N_0\left(1-\vartheta\zeta\right)\right)}{\mathcal{A}_0N_0\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}}=h_1\left(G,\lambda_G\right) \\ \zeta^* &= \frac{\mathcal{N}\phi c_T\left[-A+S\left(1-a\right)-\phi\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)+\psi\left(G-G_0\right)\right]-\mathcal{N}B^2\mathcal{A}_0N_0\vartheta k^a}{c_\zeta B^2+\mathcal{N}\phi^2c_T}=h_2\left(G,\lambda_G\right) \end{aligned}$$

The Modified Hamiltonian System is :

$$\dot{G} = \mathcal{N}\beta\omega\left[k^*\left(G,\lambda_G\right)\right]^{\alpha}-mG \tag{64}$$

$$\begin{aligned} \dot{\lambda}_G^{FB,w.m} &= \left(m+\nu\right)\lambda_G+\frac{c_T\left[-A+S\left(1-\alpha\right)-\phi\mathcal{N}\zeta\left(G\right)+\psi\left(G-G_0\right)\right]}{B} \\ &\quad \cdot\left[-\frac{\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+c_\zeta\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right) \end{aligned} \tag{65}$$

9.2 Second Case :The Model without geoengineering

9.2.1 Cooperation

We assume that $i=1,...,\mathcal{N}$ countries are involved in emissions

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i,\chi_i}\int_0^{\infty}e^{-\nu t}\left[\sum_{i=1}^{\mathcal{N}}\mathcal{A}_{0i}N_{0i}c_i-\frac{1}{2}\sum_{i=1}^{\mathcal{N}}c_TT^2\right]dt \\ v &= \rho-n-\pi \end{aligned} \tag{66}$$

subject to

$$\dot{G}=\sum_{i=1}^{\mathcal{N}}\beta\omega\left(1-\chi_i\right)^{\varepsilon}k_i^{\alpha}-mG$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \left[\left(1-\chi_i\right) k_i^{\alpha} - \left(\rho+\delta\right) k_i \right] - \\ &\quad -\sum_{i=1}^{\mathcal{N}} \frac{1}{2} c_T \left(\frac{-A + S\left(1-a\right) + \psi\left(G-G_0\right)}{B} \right)^2 - \\ &\quad +\lambda_G\left(t\right) \left[\sum_{i=1}^{\mathcal{N}} \beta\omega \left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} - mG \right] \end{aligned}$$

This implies that the optimal controls will be some functions of the (G, λ_G) or

$$k^* = \left(\frac{a\left(\lambda_G \beta\omega \left(1-\chi_i\right)^{\varepsilon} + \mathcal{A}_{0i} N_{0i}\left(1-\chi_i\right)\right)}{\mathcal{A}_{0i} N_{0i}\left(\rho+\delta\right)} \right)^{\frac{1}{1-\alpha}} = h_1\left(\lambda_G\right) \tag{67}$$

$$\chi^* = 1 - \left(-\frac{\mathcal{A}_{0i} N_{0i}}{\varepsilon \lambda_G \beta\omega} \right)^{\frac{1}{\varepsilon - 1}} = h_2\left(\lambda_G\right) \tag{68}$$

The Modified Hamiltonian System is :

$$\dot{G} = \sum_{i=1}^{\mathcal{N}} \beta\omega \left(1-\chi^*\right)^{\varepsilon} \left(k^*\right)^{\alpha} - mG$$

$$\dot{\lambda}_G^{c,w.g} = \left(m+\nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T}{B} \cdot \frac{-A + S\left(1-a\right) + \psi\left(G-G_0\right)}{B} \tag{69}$$

9.2.2 Open Loop

We assume that $i = 1, ..., \mathcal{N}$ countries are involved in emissions

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i, \chi_i} \int_0^{\infty} e^{-\nu t} \left[\mathcal{A}_{0i} N_{0i} c_i - \frac{1}{2} c_T T^2 \right] dt \\ v &= \rho - n - \pi \end{aligned} \tag{70}$$

subject to

$$\dot{G} = \beta\omega \left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} + \sum_{j \neq i}^{\mathcal{N}} \beta\omega \left(1-\bar{\chi}_j\right)^{\varepsilon} \bar{k}_j^{\alpha} - mG$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \mathcal{A}_{0i}N_{0i}\left[\left(1-\chi_i\right)k_i^{\alpha}-\left(\rho+\delta\right)k_i\right]- \\ &\quad -\frac{1}{2}c_T\left(\frac{-A+S\left(1-a\right)+\psi\left(G-G_0\right)}{B}\right)^2- \\ &\quad +\lambda_G\left(t\right)\left[\beta\omega\left(1-\chi_i\right)^{\varepsilon}k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\left(1-\bar{\chi}_j\right)^{\varepsilon}\bar{k}_j^{\alpha}-mG\right] \end{aligned}$$

The optimal controls will be some functions of the (G,λ_G) and with symmetry we have :

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G\beta\omega\left(1-\chi\right)^{\varepsilon}+\mathcal{A}_0N_0\left(1-\chi\right)\right)}{\mathcal{A}_0N_0\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}}=h_1\left(\lambda_G\right) \\ \chi^* &= 1-\left(-\frac{\mathcal{A}_0N_0}{\varepsilon\lambda_G\beta\omega}\right)^{\frac{1}{\varepsilon-1}}=h_2\left(\lambda_G\right) \end{aligned}$$

The Modified Hamiltonian System is :

$$\dot{G} = \mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(\lambda_G\right)\right]^{\alpha}-mG \tag{71}$$

$$\dot{\lambda}_G^{O.L,w.g} = \left(m+\nu\right)\lambda_G+\frac{\psi c_T\left[-A+S\left(1-a\right)+\psi\left(G-G_0\right)\right]}{B^2} \tag{72}$$

9.2.3 Feedback

We assume that $i=1,...,\mathcal{N}$ countries are involved in emissions

The solution is the maximization of global welfare or

$$\begin{aligned} W &= \max_{k_i,\chi_i}\int_0^{\infty}e^{-\nu t}\left[\mathcal{A}_{0i}N_{0i}c_i-\frac{1}{2}c_TT^2\right]dt \\ v &= \rho-n-\pi \end{aligned} \tag{73}$$

subject to

$$\dot{G}=\beta\omega\left(1-\chi_i\right)^{\varepsilon}k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon}\bar{k}_j^{\alpha}-mG$$

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \mathcal{A}_{0i}N_{0i}\left[\left(1-\chi_i\right)k_i^{\alpha}-\left(\rho+\delta\right)k_i\right]- \\ &\quad -\frac{1}{2}c_T\left(\frac{-A+S\left(1-a\right)+\psi\left(G-G_0\right)}{B}\right)^2- \\ &\quad +\lambda_G\left(t\right)\left[\beta\omega\left(1-\chi_i\right)^{\varepsilon}k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon}\bar{k}_j^{\alpha}-mG\right] \end{aligned}$$

The optimal controls will be some functions of the (G, λ_G) :

$$\begin{aligned} k^* &= \left(\frac{a\left(\lambda_G \beta \omega \left(1-\chi_i\right)^{\varepsilon} + \mathcal{A}_{0i} N_{0i}\left(1-\chi_i\right)\right)}{\mathcal{A}_{0i} N_{0i}\left(\rho+\delta\right)} \right)^{\frac{1}{1-\alpha}} = h_1\left(\lambda_G\right) \\ \chi^* &= 1 - \left(-\frac{\mathcal{A}_{0i} N_{0i}}{\varepsilon \lambda_G \beta \omega} \right)^{\frac{1}{\varepsilon - 1}} = h_2\left(\lambda_G\right) \end{aligned}$$

The Modified Hamiltonian System is :

$$\dot{G} = \mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(\lambda_G\right)\right]^{\alpha} - mG \tag{74}$$

$$\begin{aligned} \dot{\lambda}_G^{FB,w.g} &= \left(m+\nu++\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\left(\chi_j\left(G\right)\right)^{'}\right)\lambda_G \\ &\quad +\psi c_T \frac{-A+S\left(1-a\right)+\psi\left(G-G_0\right)}{B^2} \end{aligned} \tag{75}$$

9.3 Comparison

From $(59), (69)$ we have that $|\lambda_G^{w.g}| < |\lambda_G^{w.m}|$ for the model with cooperation among countries. The shadow cost of GHGs accumulation in a model without mitigation under cooperation is higher than the shadow cost of GHGs accumulation in a model without geoengineering,so the Steady State stock of GHGs accumulation will be lower in the first model than in the second and therefore the capital accumulation will be higher at an economy without any implementation of geoengineering than at an economy without any implementation of mitigation.

From $(62), (72)$ we have that $|\lambda_G^{w.g}| < |\lambda_G^{w.m}|$ for the model with Open Loop Nash Equilibrium. We can see that the results about the shadow cost of GHGs accumulation,the Steady State stock of GHGs and the capital accumulation will be exactly the same as the results in the case of cooperation among countries.

From $(65), (75)$ for a low level of GHGs accumulation we have that $-\frac{\phi(\mathcal{N}-1)}{B}\zeta'(G) < \frac{\psi}{B}$ and so $|\lambda_G^{w.g}| < |\lambda_G^{w.m}|$ for the model with Feedback Nash Equilibrium. We can see that the results about the shadow cost of GHGs accumulation,the Steady State stock of GHGs and the capital accumulation will be exactly the same as the results in the case of cooperation among countries.

The model without geoengineering has in the Steady State a higher concentration level of GHGs and a lower shadow cost of GHGs accumulation. So each country ignores the adverse affects on foreign social welfare of an additional unit of production and pollution and therefore produces too much. In an economy with only the implementation of geoengineering we have a lower level of pollution, but also a lower level of capital accumulation and production. So we can say that the current policy with only the implementation of mitigation is not as successful as the hypothetical case of the policy with only geoengineering regarding the reduction of earth's temperature and the confrontation of global warming.

Despite all these the main problem of implementation of geoengineering is the uncertainty of the consequences on the earth's climate once the scientists apply the methods of geoengineering. So the damages from geoengineering incorporate more uncertainty and this will affect decision making between mitigation and geoengineering.

10 Concluding Remarks

The efforts to deal with the problem of global warming have focused on limiting the emissions of GHGs in the atmosphere. The attempts of global or international cooperation for the limitation of the emissions of GHGs had not the expected results. In this paper we tried to combine the traditional approach to policy design of mitigation with the most recent approach of geoengineering.

If there is cooperation among countries then ,as expected, the level of accumulation of GHGs is lower than in the case of non-cooperation. However at the same time we have lower capital accumulation and production. So if the countries cooperate and apply both mitigation and geoengineering to address the problem of global warming they could significantly reduce the pollution. In the case of non-cooperation among countries the traditional result is obtained when the damage cost of geoengineering is lower than the damage cost from the increase of temperature. According to this result the shadow cost of GHGs accumulation is higher in Open Loop Nash equilibrium than in Feedback Nash equilibrium, so due to the fact that Feedback Nash equilibrium is time consistent relative to Open Loop Nash equilibrium, Feedback Nash equilibrium is a more appropriate equilibrium concept to use from a theoretical point of view than Open Loop Nash equilibrium. In the case of non-cooperation among countries we also have a new result which is obtained when the damage cost of geoengineering is higher than the damage cost from the increase of temperature. In this case we noticed that it is possible the effect of free riding to be compensated and the shadow cost of GHGs accumulation to be lower in Open Loop Nash equilibrium than in Feedback Nash equilibrium. So in this case we can say that if the damage cost of geoengineering is higher than the damage cost from the increase of temperature then Open Loop Nash equilibrium could produce more GHGs concentration relative to Feedback Nash equilibrium.

The next step was to study two extreme cases. In the first case we had the implementation of geoengineering without mitigation and in the second we had the exact opposite. We concluded that the implementation of geoengineering without mitigation is more preferable, for the limitation of global warming, than the implementation of mitigation without geoengineering. This conclusion is based on the results we had when we studied the two models, in which the steady state concentration of GHGs, capital accumulation and production in the model without mitigation are lower than those in the model without geoengineering. But these results depend on the level of the damage cost of geoengineering. We have assumed ,in addition to assumptions about damage cost of geoengineering and costs from temperature increase, that the cost of

carrying out geoengineering is lower than the mitigation cost. If the damage cost of geoengineering is extremely high then the level of geoengineering tends to zero in the case without mitigation. So if the policy maker had to choose between the implementation of geoengineering without mitigation and the opposite, he would choose geoengineering if the damage cost of geoengineering was low but if it was high then he would have no incentive for applying the methods of geoengineering. Since the damage cost of geoengineering is highly uncertain arguments in support of geoengineering need very strong qualifications.

In order to have an indication of the decisions that policy makers would make if they had to choose between the present state of limitation of emissions and the new concept of geoengineering we examined the level of mitigation for which the policy maker is willing to exchange for cost of geoengineering, while keeping his overall utility constant. We concluded that in the case of non-cooperation among countries the policy maker has higher incentive for geoengineering in contrast to mitigation and is more willing to exchange mitigation with geoengineering along the optimal trajectory. This result is based on the fact that in a non-cooperative equilibrium if mitigation expenditures increase then the policy maker will change his policy and exchange mitigation with geoengineering along the optimal trajectory; but in cooperation the policy maker will not allow a high level of GHGs accumulation in order to have more production and so he will not change his policy and exchange mitigation with geoengineering. So if the international community cannot come to an agreement then we expect at some point some nations to engage in geoengineering despite the unknown risks.

Future research will be concerned with the uncertainty of the consequences on the earth's climate by geoengineering and how this uncertainty will affect decision making between mitigation and geoengineering. Another area of further research is to consider asymmetric countries where larger countries might have different incentives in regarding mitigation and geoengineering or to study a model of space - time analysis.

11 Appendix

11.1 Optimality Conditions

11.1.1 The Model with Cooperation among countries

The generalized current value Hamiltonian function :

$$\begin{aligned} H &= \sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \left[\left(1-\chi_i-\vartheta\zeta_i\right) k_i^{\alpha} - \left(\rho+\delta\right) k_i \right] - \\ &\quad -\frac{1}{2}\sum_{i=1}^{\mathcal{N}} c_T \left(\frac{-A+S\left(1-a\right)-\phi\sum\limits_{i=1}^{\mathcal{N}}\zeta_i+\psi\left(G-G_0\right)}{B} \right)^2 - \\ &\quad -\frac{c_\zeta}{2\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\zeta_i^2 + \lambda_G\left(t\right)\left[\sum_{i=1}^{\mathcal{N}} \beta\omega\left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha} - mG \right] \end{aligned}$$

First Order Necessary Conditions for k_i, χ_i and ζ_i

$$\frac{\partial H}{\partial k_i} = 0$$

$$\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} \left[\alpha\left(1-\chi_i-\vartheta\zeta_i\right) k_i^{\alpha-1} - \left(\rho+\delta\right) \right] + \alpha\lambda_G \sum_{i=1}^{\mathcal{N}} \beta\omega\left(1-\chi_i\right)^{\varepsilon} k_i^{\alpha-1} = 0$$

$$\Longrightarrow k^* = \left(\frac{a\left(\lambda_G\beta\omega\left(1-\chi_i\right)^{\varepsilon} + \mathcal{A}_{0i}N_{0i}\left(1-\chi_i-\vartheta\zeta_i\right)\right)}{\mathcal{A}_{0i}N_{0i}\left(\rho+\delta\right)} \right)^{\frac{1}{1-\alpha}} \tag{76}$$

$$\frac{\partial H}{\partial \chi_i} \leq 0, \chi_i \geq 0$$

$$-\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} k_i^{\alpha} - \varepsilon\lambda_G \sum_{i=1}^{\mathcal{N}} \beta\omega\left(1-\chi_i\right)^{\varepsilon-1} k_i^{\alpha} \leq 0$$

$$\Longrightarrow \varepsilon\lambda_G \sum_{i=1}^{\mathcal{N}} \beta\omega\left(1-\chi_i\right)^{\varepsilon-1} = -\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} k_i^{\alpha} \text{ (interior solution)}$$

$$\Longrightarrow \chi^* = 1 - \left(-\frac{\mathcal{A}_{0i}N_{0i}}{\varepsilon\lambda_G\beta\omega} \right)^{\frac{1}{\varepsilon-1}} \tag{77}$$

$$\frac{\partial H}{\partial \zeta_i} \leq 0, \zeta_i \geq 0$$

$$-\vartheta\sum_{i=1}^{\mathcal{N}} \mathcal{A}_{0i} N_{0i} k_i^{\alpha} + \mathcal{N}\frac{\phi}{B}\sum_{i=1}^{\mathcal{N}} c_T \left(\frac{-A+S\left(1-a\right)-\phi\sum\limits_{i=1}^{\mathcal{N}}\zeta_i+\psi\left(G-G_0\right)}{B} \right) - \frac{c_\zeta}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\zeta_i \leq 0$$

$$\mathcal{N}\frac{\phi}{B}\mathcal{N}c_T \left(\frac{-A+S\left(1-a\right)-\phi\mathcal{N}\zeta_i+\psi\left(G-G_0\right)}{B} \right) - c_\zeta\zeta_i = \mathcal{N}\vartheta\mathcal{A}_{0i}N_{0i}k_i^{\alpha} \text{ (interior solution)}$$

$$\Longrightarrow \zeta^* = \frac{\mathcal{N}^2\phi c_T\left[-A+S\left(a-1\right)+\psi\left(G-G_0\right)\right]-\mathcal{N}B^2\mathcal{A}_{0i}N_{0i}\vartheta k_i^a}{c_\zeta B^2+\mathcal{N}^3\phi^2 c_T} \tag{78}$$

The Modified Hamiltonian System is :

$$\dot{G} = \frac{\partial H}{\partial \lambda_G} = \sum_{i=1}^{\mathcal{N}} \beta\omega\left(1-\chi^*\right)^{\varepsilon}\left(k^*\right)^{\alpha} - mG \tag{79}$$

$$\dot{\lambda}_G = \nu\lambda_G - \frac{\partial H}{\partial G}$$

$$\Longrightarrow \dot{\lambda}_G = \left(m+\nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T\left[-A+S\left(1-a\right)-\sum\limits_{i=1}^{\mathcal{N}}\phi\zeta^*+\psi\left(G-G_0\right)\right]}{B^2} \tag{80}$$

11.1.2 The Model with Non-Cooperation among countries (Open Loop Nash equilibrium)

The generalized current value noncooperative Hamiltonian function :

$$\begin{aligned} H^{OL} \;=\; & \mathcal{A}_{0i}N_{0i}\left[\left(1-\chi_i-\vartheta\zeta_i\right)k_i^{\alpha}-\left(\rho+\delta\right)k_i\right] \\ & -\frac{1}{2}c_T\left(\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\bar{\zeta}_j\right)+\psi\left(G-G_0\right)}{B}\right)^2 \\ & -\frac{c_\zeta}{2\mathcal{N}}\left(\zeta_i^2+\sum_{j\neq i}^{\mathcal{N}}\bar{\zeta}_j^2\right)+\lambda_G\left(t\right)\left[\beta\omega\left(1-\chi_i\right)^{\varepsilon}k_i^{\alpha}+\sum_{j\neq i}^{\mathcal{N}}\beta\omega\left(1-\bar{\chi}_j\right)^{\varepsilon}\bar{k}_j^{\alpha}-mG\right] \end{aligned}$$

First Order Necessary Conditions for k_i, χ_i and ζ_i

$$\frac{\partial H}{\partial k_i} = 0$$

$$\mathcal{A}_{0i}N_{0i}\left[\alpha\left(1-\chi_i-\vartheta\zeta_i\right)k^{\alpha-1}-\left(\rho+\delta\right)\right]+\alpha\lambda_G\beta\omega\left(1-\chi\right)^{\varepsilon}k_i^{\alpha-1}=0$$

$$\Longrightarrow k^* = \left(\frac{a\left(\lambda_G\beta\omega\left(1-\chi_i\right)^{\varepsilon}+\mathcal{A}_{0i}N_{0i}\left(1-\chi_i-\vartheta\zeta_i\right)\right)}{\mathcal{A}_{0i}N_{0i}\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}} \tag{81}$$

$$\frac{\partial H}{\partial \chi_i} \leq 0, \chi_i \geq 0$$

$$-\mathcal{A}_{0i}N_{0i}k_i^{\alpha}-\varepsilon\lambda_G\beta\omega\left(1-\chi_i\right)^{\varepsilon-1}k_i^{\alpha}\leq 0$$

$$\Longrightarrow \varepsilon\lambda_G\beta\omega\left(1-\chi_i\right)^{\varepsilon-1}k_i^{\alpha} = -\mathcal{A}_{0i}N_{0i}k_i^{\alpha} \text{ (interior solution)}$$

$$\Longrightarrow \chi^* = 1-\left(-\frac{\mathcal{A}_{0i}N_{0i}}{\varepsilon\lambda_G\beta\omega}\right)^{\frac{1}{\varepsilon-1}} \tag{82}$$

$$\frac{\partial H}{\partial \zeta_i} \leq 0, \zeta_i \geq 0$$

$$-\vartheta \mathcal{A}_{0i} N_{0i} k_i^{\alpha} + \frac{\phi}{B} c_T \left(\frac{-A + S(1-a) - \phi\left(\zeta_i + \sum_{j \neq i}^{\mathcal{N}} \bar{\zeta}_j\right) + \psi(G - G_0)}{B} \right) - \frac{c_\zeta}{\mathcal{N}} \zeta_i \leq 0$$

we impose symmetry so $\zeta_i = \bar{\zeta}_j = \zeta$ and $k_i = k$ and $\chi_i = \chi$, $\mathcal{A}_{0i} = \mathcal{A}_0, N_{0i} = N_0$ for all i.

$$\frac{\phi}{B} c_T \left(\frac{-A + S(1-a) - \phi \mathcal{N} \zeta + \psi(G - G_0)}{B} \right) - \frac{c_\zeta}{\mathcal{N}} \zeta = \vartheta \mathcal{A}_0 N_0 k^{\alpha} \text{ (interior solution)}$$

$$\Longrightarrow \zeta^* = \frac{\mathcal{N} \phi c_T \left[-A + S(1-a) + \psi(G - G_0)\right] - \mathcal{N} B^2 \mathcal{A}_0 N_0 \vartheta k^a}{c_\zeta B^2 + \mathcal{N}^2 \phi^2 c_T} \tag{83}$$

The Modified Hamiltonian System is :

$$\dot{G} = \frac{\partial H}{\partial \lambda_G} = \mathcal{N} \beta \omega \left(1 - \chi^*\right)^{\varepsilon} \left(k^*\right)^{\alpha} - mG \tag{84}$$

$$\dot{\lambda}_G = \nu \lambda_G - \frac{\partial H}{\partial G}$$

$$\Longrightarrow \dot{\lambda}_G = (m + \nu) \lambda_G + \frac{\psi c_T \left[-A + S(1-a) - \sum_{i=1}^{\mathcal{N}} \phi \zeta^* + \psi(G - G_0)\right]}{B^2} \tag{85}$$

11.1.3 The Model with Non-Cooperation among countries (Feedback Nash equilibrium)

The generalized current value noncooperative Hamiltonian function :

$$\begin{aligned} H^F = \; & \mathcal{A}_{0i} N_{0i} \left[\left(1 - \chi_i - \vartheta \zeta_i\right) k_i^{\alpha} - (\rho + \delta) k_i\right] \\ & - \frac{1}{2} c_T \left(\frac{-A + S(1-a) - \phi\left(\zeta_i + \sum_{j \neq i}^{\mathcal{N}} \left(\zeta_j(G)\right)\right) + \psi(G - G_0)}{B} \right)^2 \\ & - \frac{c_\zeta}{2\mathcal{N}} \left(\zeta_i^2 + \sum_{j \neq i}^{\mathcal{N}} \left(\zeta_j(G)\right)^2 \right) + \lambda_G(t) \left[\beta \omega \left(1 - \chi_i\right)^{\varepsilon} k_i^{\alpha} + \sum_{j \neq i}^{\mathcal{N}} \beta \omega \left(1 - \chi_j(G)\right)^{\varepsilon} \bar{k}_j^{\alpha} - mG \right] \end{aligned}$$

This implies that the optimal controls will be some functions of the (G, λ_G)
First Order Necessary Conditions for k_i, χ_i and ζ_i

$$\frac{\partial H}{\partial k_i} = 0$$

$$\mathcal{A}_{0i} N_{0i} \left[\alpha \left(1 - \chi_i - \vartheta \zeta_i\right) k_i^{\alpha - 1} - (\rho + \delta)\right] + \alpha \lambda_G \beta \omega \left(1 - \chi_i\right)^{\varepsilon} k_i^{\alpha - 1} = 0$$

$$\Longrightarrow k^{*}=\left(\frac{a\left(\lambda_{G}\beta\omega\left(1-\chi_{i}\right)^{\varepsilon}+\mathcal{A}_{0i}N_{0i}\left(1-\chi_{i}-\vartheta\zeta_{i}\right)\right)}{\mathcal{A}_{0i}N_{0i}\left(\rho+\delta\right)}\right)^{\frac{1}{1-\alpha}} \tag{86}$$

$$\frac{\partial H}{\partial\chi_{i}}\leq 0,\chi_{i}\geq 0$$

$$-\mathcal{A}_{0i}N_{0i}k_{i}^{\alpha}-\varepsilon\lambda_{G}\beta\omega\left(1-\chi_{i}\right)^{\varepsilon-1}k_{i}^{\alpha}\leq 0$$

$$\Longrightarrow\varepsilon\lambda_{G}\beta\omega\left(1-\chi_{i}\right)^{\varepsilon-1}=-\mathcal{A}_{0i}N_{0i}k_{i}^{\alpha}\ \text{(interior solution)}$$

$$\Longrightarrow\chi^{*}=1-\left(-\frac{\mathcal{A}_{0i}N_{0i}}{\varepsilon\lambda_{G}\beta\omega}\right)^{\frac{1}{\varepsilon-1}} \tag{87}$$

$$\frac{\partial H}{\partial\zeta_{i}}\leq 0,\zeta_{i}\geq 0$$

$$-\vartheta\mathcal{A}_{0i}N_{0i}k_{i}^{\alpha}+\frac{\phi}{B}c_{T}\left(\frac{-A+S\left(1-a\right)-\phi\left(\zeta_{i}+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_{j}\left(G\right)\right)\right)+\psi\left(G-G_{0}\right)}{B}\right)-\frac{c_{\zeta}}{\mathcal{N}}\zeta_{i}\leq 0$$

$$\mathcal{N}\frac{\phi}{B}c_{T}\left(\frac{-A+S\left(1-a\right)-\phi\left(\zeta_{i}+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_{j}\left(G\right)\right)\right)+\psi\left(G-G_{0}\right)}{B}\right)-c_{\zeta}\zeta_{i}=\mathcal{N}\vartheta\mathcal{A}_{0i}N_{0i}k_{i}^{\alpha}\ \text{(interior solution)}$$

$$\Longrightarrow\zeta^{*}=\frac{\mathcal{N}\phi c_{T}\left[-A+S\left(1-a\right)-\phi\sum_{j\neq i}^{\mathcal{N}}\zeta_{j}\left(G\right)+\psi\left(G-G_{0}\right)\right]-\mathcal{N}B^{2}\mathcal{A}_{0i}N_{0i}\vartheta k_{i}^{a}}{c_{\zeta}B^{2}+\mathcal{N}\phi^{2}c_{T}} \tag{88}$$

The Modified Hamiltonian System characterizing the feedback Nash equilibrium (FBNE) under symmetry implies

$$\dot{G}=\frac{\partial H}{\partial\lambda_{G}}=\mathcal{N}\beta\omega\left(1-\chi^{*}\right)^{\varepsilon}\left(k^{*}\right)^{\alpha}-mG \tag{89}$$

$$\dot{\lambda}_{G}=\nu\lambda_{G}-\frac{\partial H}{\partial G}$$

$$\begin{aligned}\dot{\lambda}_{G}^{F} &= \left(m+\nu\right)\lambda_{G}+c_{T}T\left[\frac{\left(-\right)\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+c_{\zeta}\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right)\\ &+\varepsilon\lambda_{G}\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_{j}\left(G\right)\right)^{\varepsilon-1}\bar{k}_{j}^{\alpha}\left(\chi_{j}\left(G\right)\right)'\end{aligned} \tag{90}$$

11.2 Conditions for Stability

11.2.1 The Model with Cooperation among countries

$$\dot{G} = \mathcal{N}\beta\omega\left(1 - h_2\left(\lambda_G\right)\right)^{\varepsilon}\left[h_1\left(G, \lambda_G\right)\right]^{\alpha} - mG = 0 \tag{91}$$

$$\Longrightarrow H_{\lambda_G} = \mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha} - mG$$

$$\dot{\lambda}_G^c = \left(m + \nu\right)\lambda_G + \frac{\mathcal{N}\psi c_T}{B} \cdot \frac{-A + S\left(1 - a\right) - \phi\mathcal{N}h_3\left(G, \lambda_G\right) + \psi\left(G - G_0\right)}{B} = 0 \tag{92}$$

$$\Longrightarrow H_{_G} = m\lambda_G + \frac{\mathcal{N}\psi c_T}{B} \cdot \frac{-A + S\left(1 - a\right) - \phi\mathcal{N}\zeta^*\left(G, \lambda_G\right) + \psi\left(G - G_0\right)}{B}$$

For the Jacobian matrix:

$$\frac{\partial H_{\lambda_G}}{\partial G} = \alpha\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial G} - m$$

$$\begin{aligned}\frac{\partial H_{\lambda_G}}{\partial \lambda_G} &= -\varepsilon\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon-1}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha}\frac{\partial \chi^*}{\partial \lambda_G} + \\ &\quad + \alpha\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial \lambda_G}\end{aligned}$$

$$\begin{aligned}\Longrightarrow \quad \frac{\partial H_{\lambda_G}}{\partial \lambda_G} &= -\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon-1}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1} \cdot \\ &\quad \cdot \left[\varepsilon k^*\left(G, \lambda_G\right)\frac{\partial \chi^*}{\partial \lambda_G} + \alpha\left(\chi^*\left(\lambda_G\right) - 1\right)\frac{\partial k^*}{\partial \lambda_G}\right]\end{aligned}$$

$$\frac{\partial H_{_G}}{\partial G} = \frac{\mathcal{N}\psi c_T}{B^2} \cdot \left(\psi - \phi\mathcal{N}\frac{\partial \zeta^*}{\partial G}\right)$$

$$\frac{\partial H_{_G}}{\partial \lambda_G} = m - \frac{\phi\mathcal{N}^2\psi c_T}{B^2} \cdot \frac{\partial \zeta^*}{\partial \lambda_G}$$

$$j = \begin{matrix} \frac{\partial H_{\lambda_G}}{\partial G} & \frac{\partial H_{\lambda_G}}{\partial \lambda_G} \\ \frac{\partial H_{_G}}{\partial G} & \frac{\partial H_{_G}}{\partial \lambda_G} \end{matrix}$$

$$\det\left(j\right) = \frac{\partial H_{\lambda_G}}{\partial G} \cdot \frac{\partial H_{_G}}{\partial \lambda_G} - \frac{\partial H_{\lambda_G}}{\partial \lambda_G} \cdot \frac{\partial H_{_G}}{\partial G}$$

11.2.2 The Model with Non-Cooperation among countries (Open Loop Nash equilibrium)

$$\dot{G} = \mathcal{N}\beta\omega\left(1 - h_2\left(\lambda_G\right)\right)^{\varepsilon}\left[h_1\left(G, \lambda_G\right)\right]^{\alpha} - mG = 0 \tag{93}$$
$$\Longrightarrow H_{\lambda_G} = \mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha} - mG$$

$$\dot{\lambda}_G^{O.L} = \left(m + \nu\right)\lambda_G + \frac{\psi c_T}{B} \cdot \frac{-A + S\left(1 - a\right) - \phi\mathcal{N}h_3\left(G, \lambda_G\right) + \psi\left(G - G_0\right)}{B} = 0 \tag{94}$$
$$\Longrightarrow H_{_G} = m\lambda_G + \frac{\psi c_T}{B} \cdot \frac{-A + S\left(1 - a\right) - \phi\mathcal{N}\zeta^*\left(G, \lambda_G\right) + \psi\left(G - G_0\right)}{B}$$

For the Jacobian matrix:

$$\frac{\partial H_{\lambda_G}}{\partial G} = \alpha\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial G} - m$$

$$\begin{aligned}\frac{\partial H_{\lambda_G}}{\partial \lambda_G} &= -\varepsilon\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon-1}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha}\frac{\partial \chi^*}{\partial \lambda_G} + \\ &\quad + \alpha\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial \lambda_G}\end{aligned}$$

$$\begin{aligned}\Longrightarrow \quad \frac{\partial H_{\lambda_G}}{\partial \lambda_G} &= -\mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon-1}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha-1} \cdot \\ &\quad \cdot \left[\varepsilon k^*\left(G, \lambda_G\right)\frac{\partial \chi^*}{\partial \lambda_G} + \alpha\left(\chi^*\left(\lambda_G\right) - 1\right)\frac{\partial k^*}{\partial \lambda_G}\right]\end{aligned}$$

$$\frac{\partial H_{_G}}{\partial G} = \frac{\psi c_T}{B^2} \cdot \left(\psi - \phi\mathcal{N}\frac{\partial \zeta^*}{\partial G}\right)$$

$$\frac{\partial H_{_G}}{\partial \lambda_G} = m - \frac{\phi\mathcal{N}\psi c_T}{B^2} \cdot \frac{\partial \zeta^*}{\partial \lambda_G}$$

$$j = \begin{matrix} \frac{\partial H_{\lambda_G}}{\partial G} & \frac{\partial H_{\lambda_G}}{\partial \lambda_G} \\ \frac{\partial H_{_G}}{\partial G} & \frac{\partial H_{_G}}{\partial \lambda_G} \end{matrix}$$

$$\det\left(j\right) = \frac{\partial H_{\lambda_G}}{\partial G} \cdot \frac{\partial H_{_G}}{\partial \lambda_G} - \frac{\partial H_{\lambda_G}}{\partial \lambda_G} \cdot \frac{\partial H_{_G}}{\partial G}$$

11.2.3 The Model with Non-Cooperation among countries (Feedback Nash equilibrium)

$$\dot{G} = \mathcal{N}\beta\omega\left(1 - h_2\left(\lambda_G\right)\right)^{\varepsilon}\left[h_1\left(G, \lambda_G\right)\right]^{\alpha} - mG = 0 \tag{95}$$
$$\Longrightarrow H_{\lambda_G} = \mathcal{N}\beta\omega\left(1 - \chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G, \lambda_G\right)\right]^{\alpha} - mG$$

$$\dot{\lambda}_G^F = \left(m+\nu+\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\left(\chi_j\left(G\right)\right)'\right)\lambda_G \tag{96}$$

$$+c_T\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)\right)+\psi\left(G-G_0\right)}{B} \tag{97}$$

$$\cdot\left[-\frac{\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right) \tag{98}$$

$$\Longrightarrow\quad H_{_G}=\left(m+\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\left(\chi_j\left(G\right)\right)'\right)\lambda_G$$

$$+c_T\frac{-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)\right)+\psi\left(G-G_0\right)}{B}$$

$$\cdot\left[-\frac{\phi\left(\mathcal{N}-1\right)}{B}\zeta'\left(G\right)+\frac{\psi}{B}\right]+c_\zeta\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\zeta\left(G\right)\zeta'\left(G\right)$$

For the Jacobian matrix:

$$\frac{\partial H_{\lambda_G}}{\partial G}=\alpha\mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G,\lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial G}-m$$

$$\frac{\partial H_{\lambda_G}}{\partial\lambda_G}=-\varepsilon\mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon-1}\left[k^*\left(G,\lambda_G\right)\right]^{\alpha}\frac{\partial\chi^*}{\partial\lambda_G}+$$

$$+\alpha\mathcal{N}\beta\omega\left(1-\chi^*\left(\lambda_G\right)\right)^{\varepsilon}\left[k^*\left(G,\lambda_G\right)\right]^{\alpha-1}\frac{\partial k^*}{\partial\lambda_G}$$

$$\frac{\partial H_{_G}}{\partial G}=\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\lambda_G\left[-(\varepsilon-1)[(\chi_j(G))']^2\left(1-\chi_j\left(G\right)\right)^{-1}+\left[\chi^*\left(G\right)''\right]\right]$$

$$+\frac{c_T}{B^2}\left[\psi-\phi(N-1)\zeta'\left(G\right)\right]^2-$$

$$-\frac{(N-1)\phi\zeta''(G)c_T\left[-A+S\left(1-a\right)-\phi\left(\zeta_i+\sum_{j\neq i}^{\mathcal{N}}\left(\zeta_j\left(G\right)\right)\right)+\psi\left(G-G_0\right)\right]}{B^2}$$

$$+c_\zeta\frac{\left(\mathcal{N}-1\right)}{\mathcal{N}}\left[\left[\zeta'\left(G\right)\right]^2+\zeta\left(G\right)\zeta''\left(G\right)\right]$$

Assumptions : *i)* $\zeta''\left(G\right),\chi''\left(G\right)<0$, *ii)* $\left[\zeta'\left(G\right)\right]^2+\zeta\left(G\right)\zeta''\left(G\right)>0$ *and iii)* low level of current accumulation of $GHGs$

$$\frac{\partial H_{_G}}{\partial\lambda_G}=m+\varepsilon\left(\mathcal{N}-1\right)\beta\omega\left(1-\chi_j\left(G\right)\right)^{\varepsilon-1}\bar{k}_j^{\alpha}\left(\chi_j\left(G\right)\right)'$$

$$j=\begin{matrix}\frac{\partial H_{\lambda_G}}{\partial G} & \frac{\partial H_{\lambda_G}}{\partial\lambda_G}\\ \frac{\partial H_{_G}}{\partial G} & \frac{\partial H_{_G}}{\partial\lambda_G}\end{matrix}$$

$$\det\left(j\right)=\frac{\partial H_{\lambda_G}}{\partial G}\cdot\frac{\partial H_{_G}}{\partial\lambda_G}-\frac{\partial H_{\lambda_G}}{\partial\lambda_G}\cdot\frac{\partial H_{_G}}{\partial G}$$

12 References

[1] G.R. North, Analytical solution to a simple climate model with diffusive heat transport, Journal of the Atmospheric Sciences 32 (7) (1975) 1301-1307

[2] G.R. North, Theory of energy-balance climate models, Journal of the Atmospheric Sciences 32 (11) (1975) 2033-2043

[3] G.R. North, Energy balance climate models, Reviews of Geophysics and Space Physics 19 (1) (1981) 91-121

[4] W.D. Nordhaus, Economic aspects of global warming in a post-Copenhagen environment, PNAS 107 (26) (2010)

[5] A. Robock, 20 reasons why geoengineering may be a bad idea, Bulletin of the Atomic Scientists 64 (2008) 14-18

[6] S. Barrett, The incredible economics of geoengineering, Environmental and resource economics 39 (1) (2008) 45-54

[7] W. Brock, A. Xepapadeas, Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control, Journal of Economic Dynamics and Control 32 (9) (2008) 2745–2787

[8] L. C. Evans, An introduction to mathematical optimality control theory, 1983

[9] M.I. Kamien, N.L. Schwartz, Dynamic Optimization : the calculus of variations and optimal control in economics and management, 1981

[10] W. Brock, G. Engstrőm, A. Xepapadeas, Energy balance climate models and the economics of climate change, Beijer discussion paper 226, The Beijer Institute of Ecological Economics, 2011

[11] T. Roughgarden, S.H. Schneider, Climate change policy: quantifying uncertainties for damages and optimal carbon taxes, Journal of Energy Policy 27 (7) (1999) 415-429

[12] J. P. Bruce , H. Yi, E. F. Haites , Economic and social dimensions of climate change, Cambridge University Press, 1996

[13] W. R. Cline, The economics of global warming, Peterson Institute, 1992

[14] P.A Simmons, D.H Griffel, Continuous versus discontinuous albedo representations in a simple diffusive climate model, Climate Dynamics 3 (1) (1988) 35–39

[15] J.A. Coakley, B.A. Wielicki, Testing Energy balance climate models, Journal of the Atmospheric Sciences 36 (1979) 2031-2039

[16] J.A. Coakley, A study of climate sensitivity using a simple Energy balance climate model, Journal of the Atmospheric Sciences 36 (1979) 260-269

[17] M.I. Budyko, On present - day climatic changes, Tellus 29 (3) (1977) 193-204

[18] G.R. North, L. Howard, D. Pollard, B. Wielicki, Variational formulation of Budyko - Sellers climate models, Journal of the Atmospheric Sciences 36 (2) (1979) 255-259

[19] W.D. Nordhaus, The architecture of climate economics: Designing a global agreement on global warming,Bulletin of the Atomic Scientists 67 (1) (2011) 9–18

[20] W.D. Nordhaus, J. Boyer, Warming the world: economic models of global warming, MIT Press, 2003

[21] W.D. Nordhaus, A question of balance: economic modeling of global warming, Yale University Press, 2007

[22] C.O. Criado, S. Valente, T. Stengos, Growth and pollution convergence: Theory and evidence, Journal of Environmental Economics and Management 62 (2) (2011) 199-214

[23] A. Xepapadeas, Environmental policy design and dynamic nonpoint-source pollution, Journal of Environmental Economics and Management 23 (1) (1992) 22-39

[24] F. van der Ploeg, A. J. De Zeeuw, International aspects of pollution control, Environmental and Resource Economics 2 (2) (1992) 117-139

[25] J.B. Moreno-Cruz, Climate policy under uncertainty: a case for geo-engineering, World Congress of Environmental and Resource Economics, 2010

[26] K. Gramstad,S. Tjotta, Climate engineering: cost benefit and beyond, 2010

[27] M.L. Weitzman, What is the "damages function" for global warming-and what difference might it make?, Journal of Climate Change Economics 1 (1) (2010) 57-69

[28] O. Tahvonen, J. Kuuluvainen, Economic growth, pollution, and renewable resources, Journal of Environmental Economics and Management 24 (2) (1993) 101-118

[29] R. Boucekkine, C. Camacho, B. Zou, Bridging the gap between growth theory and the new economic geography: The spatial Ramsey model, Journal of Macroeconomic Dynamics 13 (1) (2010)